Inductive Inductive Types

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Motivation

- Has been implemented in Agda
- Has been used to study type theory within itself
- This paper verifies consistency

What is an inductive-inductive type?

An inductive type $A : Set$ together with **type indexed family** $B : A \to Set$

```
Inductive A : Set :=
| ..
mutual B : A -> Set :=
| ..
```

Inductive-Inductive buildings

$$\begin{aligned}
\mathsf{ground} &: \mathsf{Platform}, \\
\mathsf{extension} &: ((p : \mathsf{Platform}) \times \mathsf{Building}(p)) \to \mathsf{Platform}, \\
\mathsf{onTop} &: (p : \mathsf{Platform}) \to \mathsf{Building}(p), \\
\mathsf{hangingUnder} &: ((p : \mathsf{Platform}) \times (b : \mathsf{Building}(p))) \to \mathsf{Building}(\mathsf{extension}(\langle p, b \rangle))
\end{aligned}$$

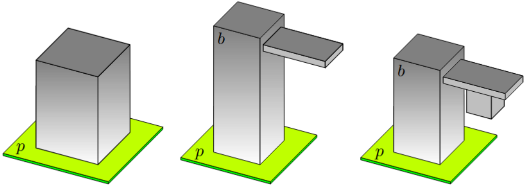

Fig. 1. onTop(p), extension($\langle p, b\rangle$) and hangingUnder($\langle p, b\rangle$).

Simultaneous inductive to Inductive-Inductive

Simultaneous inductive

$$\text{intro}_A : \Phi_A(A, B) \to A \qquad \text{intro}_B : \Phi_B(A, B) \to B$$

Inductive-inductive

$$\text{intro}_A : \Phi_A(A, B) \to A \qquad \text{intro}_B : (a : \Phi_B(A, B)) \to B(i_{A,B}(a))$$

Strictly Positive

Remember from ~~yesterday~~ last week that we had

$$\mathrm{intro} : \Phi(A) \to A$$

where the functor Φ was constructed as follows:

- No premises: $\Phi(A) = \mathbf{1}$
- Non-inductive premise: $\Phi(A) = (x : K) \times \Psi_x(A)$
- Inductive premise: $\Phi(A) = (K \to A) \times \Psi(A)$

Strictly Positive Operators

If we then move to defining two sets, we get

$$\text{intro}_A : \Phi_A(A, B) \to A \qquad \text{intro}_B : \Phi_B(A, B) \to B$$

- No premises: $\Phi(A, B) = \mathbf{1}$
- Non-inductive premise: $\Phi(A, B) = (x \colon K) \times \Psi_x(A, B)$
- Premise inductive in A: $\Phi(A, B) = (K \to A) \times \Psi(A, B)$
- Premise inductive in B: $\Phi(A, B) = (K \to B) \times \Psi(A, B)$

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* : This Ψ_f is only allowed to depend on $f : K \to A$ for indices of B

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- Premise inductive in B:
  $\Phi(A, B) = (f : ((x : K) \to B(i_{A,B}(x)))) \times \Psi_F(A, B)*$

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Axiomatisation using coding

$$SP_A : \text{Type} \qquad SP_B : \text{Type}$$

together with

$$\text{Arg}_A \qquad \text{Arg}_B$$

SP_A: Formation Rule

$$\frac{A_{ref} : \text{Set} \qquad B_{ref} : \text{Set}}{\text{SP}_\text{A}(A_{ref}, B_{ref}) : \text{Type}}$$

Eventually, we only want to look at codes that don't already have any elements:
$\text{SP'}_\text{A} := \text{SP}_\text{A}(\mathbf{0}, \mathbf{0})$

SP_A: Introduction Rules

$$\frac{}{\mathrm{nil_A} : \mathrm{SP_A}(A_{ref}, B_{ref})}$$

Representing a trivial constructor

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Representing a trivial constructor

$$\frac{K : \mathrm{Set} \qquad \gamma : K \to \mathrm{SP}_\mathrm{A}(A_{ref}, B_{ref})}{\mathrm{nonind}(K, \gamma) : \mathrm{SP}_\mathrm{A}(A_{ref}, B_{ref})}$$

Representing a constructor with a non-inductive argument

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Representing a constructor with a non-inductive argument

$$\frac{K : \mathrm{Set} \qquad \gamma : \mathrm{SP}_\mathrm{A}(A_{ref} + K, B_{ref})}{\text{A-ind}(K, \gamma) : \mathrm{SP}_\mathrm{A}(A_{ref}, B_{ref})}$$

Representing a constructor with an A-inductive argument

SP_A: Introduction Rules (cont)

$$\frac{K : \text{Set} \qquad h_{index} : K \to A_{ref} \qquad \gamma : \text{SP}_\text{A}(A_{ref}, B_{ref} + K)}{\text{B-ind}(K, h_{index}, \gamma) : \text{SP}_\text{A}(A_{ref}, B_{ref})}$$

Representing a constructor with a B-inductive argument

Example

If we look at the following constructor:

$$\text{extension} : ((p : \text{Platform}) \times \text{Building}(p)) \to \text{Platform}$$

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Then this rule would have following code:

$$\gamma_{ext} = \text{A-ind}(\mathbf{1}, \text{B-ind}(\mathbf{1}, \lambda * .\hat{p}, \text{nil}_\text{A}))$$

Where then $\gamma_{ext} : \text{SP'}_\text{A} = \text{SP}_\text{A}(\mathbf{0}, \mathbf{0})$, and $\hat{p} = inr(*)$ is the element representing the "induction hypothesis"

$\mathrm{Arg_A}$: Formation Rule

$$\frac{\begin{array}{c} A_{ref}, B_{ref} : \mathrm{Set} \\ \gamma : \mathrm{SP_A}(A_{ref}, B_{ref}) \end{array} \qquad \begin{array}{c} A : \mathrm{Set} \\ B : A \to \mathrm{Set} \end{array} \qquad \begin{array}{c} \mathrm{rep_A} : A_{ref} \to A \\ \mathrm{rep_{index}} : B_{ref} \to A \\ \mathrm{rep_B} : (x : B_{ref}) \to B(\mathrm{rep_{index}}(x)) \end{array}}{\mathrm{Arg_A}(A_{ref}, B_{ref}, \gamma, A, B, \mathrm{rep_A}, \mathrm{rep_{index}}, \mathrm{rep_B}) : Set}$$

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γ represents a constructor, which can make use of the elements represented by codes in A_{ref} and B_{ref}

$\mathrm{Arg}_{\mathrm{A}}$: Formation Rule

$$\frac{\begin{array}{c} A_{ref}, B_{ref} : \mathrm{Set} \\ \gamma : \mathrm{SP}_{\mathrm{A}}(A_{ref}, B_{ref}) \end{array} \qquad \begin{array}{c} A : \mathrm{Set} \\ B : A \to \mathrm{Set} \end{array} \qquad \begin{array}{c} \mathrm{rep}_{\mathrm{A}} : A_{ref} \to A \\ \mathrm{rep}_{\mathrm{index}} : B_{ref} \to A \\ \mathrm{rep}_{\mathrm{B}} : (x : B_{ref}) \to B(\mathrm{rep}_{\mathrm{index}}(x)) \end{array}}{\mathrm{Arg}_{\mathrm{A}}(A_{ref}, B_{ref}, \gamma, A, B, \mathrm{rep}_{\mathrm{A}}, \mathrm{rep}_{\mathrm{index}}, \mathrm{rep}_{\mathrm{B}}) : Set}$$

Since A and B are yet to be defined, these input sets are allowed to be arbitrary for now

Arg_A: Formation Rule

$$\frac{\begin{array}{c} A_{ref}, B_{ref} : \mathrm{Set} \\ \gamma : \mathrm{SP_A}(A_{ref}, B_{ref}) \end{array} \qquad \begin{array}{c} A : \mathrm{Set} \\ B : A \to \mathrm{Set} \end{array} \qquad \begin{array}{c} \mathrm{rep_A} : A_{ref} \to A \\ \mathrm{rep_{index}} : B_{ref} \to A \\ \mathrm{rep_B} : (x : B_{ref}) \to B(\mathrm{rep_{index}}(x)) \end{array}}{\mathrm{Arg_A}(A_{ref}, B_{ref}, \gamma, A, B, \mathrm{rep_A}, \mathrm{rep_{index}}, \mathrm{rep_B}) : Set}$$

The various rep functions map elements to their real counterparts

Arg_A: Formation Rule

$$\frac{\begin{array}{c} A_{ref}, B_{ref} : \mathrm{Set} \\ \gamma : \mathrm{SP}_\mathrm{A}(A_{ref}, B_{ref}) \end{array} \qquad \begin{array}{c} A : \mathrm{Set} \\ B : A \to \mathrm{Set} \end{array} \qquad \begin{array}{c} \mathrm{rep}_\mathrm{A} : A_{ref} \to A \\ \mathrm{rep}_\mathrm{index} : B_{ref} \to A \\ \mathrm{rep}_\mathrm{B} : (x : B_{ref}) \to B(\mathrm{rep}_\mathrm{index}(x)) \end{array}}{\mathrm{Arg}_\mathrm{A}(A_{ref}, B_{ref}, \gamma, A, B, \mathrm{rep}_\mathrm{A}, \mathrm{rep}_\mathrm{index}, \mathrm{rep}_\mathrm{B}) : Set}$$

The code γ represents a constructor. Arg_A gives the domain of that constructor.

Another definition: Arg'_A

We are mostly interested in the case where $A_{ref} = B_{ref} = \mathbf{0}$, in that case:

- $\gamma : SP'_A$
- $rep_A : \mathbf{0} \to A$
- $rep_{index} : \mathbf{0} \to A$
- $rep_B : (x : \mathbf{0}) \to B(rep_{index}(x))$

Since their types already determines our choices for these functions, we define:

$$Arg'_A(\gamma, A, B) := Arg_A(\mathbf{0}, \mathbf{0}, \gamma, A, B, !_A, !_A, !_{B \circ !_A})$$

Arg_A

The code nil_A represents a constructor with no argument, and as we saw earlier, the domain for that constructor is $\mathbf{1}$

$$\mathrm{Arg}_\mathrm{A}(A_{ref}, B_{ref}, \mathrm{nil}_\mathrm{A}, A, B, \mathrm{rep}_\mathrm{A}, \mathrm{rep}_\mathrm{index}, \mathrm{rep}_\mathrm{B}) = \mathbf{1}$$

The code $\mathrm{nonind}(K, \gamma)$ represents a constructor with a non-inductive argument

$$\mathrm{Arg}_\mathrm{A}(A_{ref}, B_{ref}, \mathrm{nonind}(K, \gamma), A, B, \mathrm{rep}_\mathrm{A}, \mathrm{rep}_\mathrm{index}, \mathrm{rep}_\mathrm{B}) = (k : K) \times \mathrm{Arg}_\mathrm{A}(\ldots, \gamma(k), \ldots)$$

Arg_A

The code $\text{A-ind}(K, \gamma)$ represents a constructor with an A-inductive argument

$$\text{Arg}_\text{A}(A_{ref}, B_{ref}, \text{A-ind}(K, \gamma), A, B, \text{rep}_\text{A}, \text{rep}_\text{index}, \text{rep}_\text{B}) = (j : K \to A) \times \text{Arg}_\text{A}(\ldots, \gamma(k), \ldots)$$

Arg_A

And $\text{B-ind}(K, h_{\text{index}}, \gamma)$ one with a B-inductive argument

$$
\begin{aligned}
ArgA(A_{ref}, B_{ref}, \text{B-ind}(K, h_{\text{index}}, \gamma), A, B, \text{rep}_\text{A}, \text{rep}_{\text{index}}, \text{rep}_\text{B}) = \\
(j : (k : K) \to B((\text{rep}_\text{A} \circ h_{\text{index}})(k))) \\
\times \text{Arg}_\text{A}(\ldots, B_{ref} + K, \gamma(k), \ldots, \text{rep}_{\text{index}} \sqcup (\text{rep}_\text{A} \circ h_{\text{index}}), \text{rep}_\text{B} \sqcup j)
\end{aligned}
$$

Example

If we go back to our example from earlier with extension, it had the following code:

$$\gamma_{ext} = \text{A-ind}(\mathbf{1}, \text{B-ind}(\mathbf{1}, \lambda * .\hat{p}, \text{nil}_\text{A}))$$

It would the following Arg'$_\text{A}$:
$\text{Arg'}_\text{A}(\gamma_{ext}, \text{Platform}, \text{Building}) = (p : \mathbf{1} \to \text{Platform}) \times \mathbf{1} \to \text{Building}(p(*)) \times \mathbf{1}$
$\text{Arg'}_\text{A}(\gamma_{ext}, \text{Platform}, \text{Building}) = (p : \text{Platform}) \times \text{Building}(p)$

Motivation

- We now have representations for (eventual) elements of A and B, and we can reference those representations

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- We might want to reference a *constructor* of A as an index for B, but such a constructor will need arguments
- We need to represent an element of $\text{Arg'}_{\text{A}}(\gamma, A, B)$

Intuitively, we might want to construct $\text{Arg'}_{\text{A}}(\gamma, A_{ref}, B_{ref})$ and then use elements from there as representations.

But: A_{ref} and B_{ref} are not quite of the right form yet

The Idea

We will construct:

- $\overline{A_{ref}} : Set$
- $\overline{B_{ref}} : \overline{A_{ref}} \to Set$
- $\overline{\mathrm{rep}_\mathrm{A}} : \overline{A_{ref}} \to A$
- $\overline{\mathrm{rep}_\mathrm{B}} : (x : \overline{A_{ref}}) \to \overline{B_{ref}}(x) \to B(\overline{\mathrm{rep}_\mathrm{A}}(x))$

From these we will then get a function

$$\mathrm{lift}'(\overline{\mathrm{rep}_\mathrm{A}}, \overline{\mathrm{rep}_\mathrm{A}}) : \mathrm{Arg'}_\mathrm{A}(\gamma, \overline{A_{ref}}, \overline{B_{ref}}) \to \mathrm{Arg'}_\mathrm{A}(\gamma, A, B)$$

A_{ref}

- A_{ref} : Everything we need to represent A
- B_{ref} : Everything we need to represent B
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- $\overline{A_{ref}}$: Everything that *actually* represents an a in A
  - So including those elements from B_{ref}
- $\overline{A_{ref}} := A_{ref} + B_{ref}$.

B_{ref}

- If $\bar{a}$ from $\overline{A_{ref}}$ represents a from A, then elements from $\overline{B_{ref}}(\bar{a})$ should represent elements from $B(a)$
- If $\bar{a}$ is from $\overline{A_{ref}}$ then it is either from A_{ref} or from B_{ref}

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- If it is from B_{ref} then we know one element: $\mathrm{rep}_{\mathrm{B}}(\bar{a})$

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- If it is from B_{ref} then we know one element: $\mathrm{rep_B}(\bar{a})$
- $\overline{B_{ref}} := (\lambda x.\mathbf{0}) \sqcup (\lambda x.\mathbf{1})$

$\overline{\mathrm{rep}_\mathrm{A}}$

We define:

- $\overline{\mathrm{rep}_\mathrm{A}} : \overline{A_{ref}} \to A = (A_{ref} + B_{ref}) \to A$

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We define:

- $\overline{\mathrm{rep}_\mathrm{A}} : \overline{A_{ref}} \to A = (A_{ref} + B_{ref}) \to A$
- How to map those to the elements of A they represent we already know:
- $\overline{\mathrm{rep}_\mathrm{A}} := \mathrm{rep}_\mathrm{A} \sqcup \mathrm{rep}_\mathrm{index}$

$\overline{\mathrm{rep}_B}$

- $\overline{\mathrm{rep}_b} : (x : \overline{A_{ref}}) \to \overline{B_{ref}}(x) \to B(\overline{\mathrm{rep}_A}(x))$

$\overline{\mathsf{rep}_\mathsf{B}}$

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- If x comes from B_{ref} then $\overline{B_{ref}}(x) = \mathbf{1}$ and we need to map that element to the one element we know exists
- $\overline{\mathrm{rep_b}} := (\lambda x.!_{B \circ !_A}) \sqcup (\lambda x : *.\mathrm{rep_B}(x))$

lift

If we have $g : A \to A^*$ and $g' : (x : A) \to B(x) \to B^*(g(x))$ then we can also construct:

$$\text{lift}'(g, g') : \text{Arg'}_\text{A}(\gamma, A, B) \to \text{Arg'}_\text{A}(\gamma, A^*, B^*)$$

We skip the proof for time reasons

Using the lift function

We now give the following two definitions

- $\overline{\text{arg}_A}(\gamma, A_{ref}, B_{ref}) := \text{Arg'}_\text{A}(\gamma, \overline{A_{ref}}, \overline{B_{ref}})$
- $\overline{\text{lift}}(\text{rep}_\text{A}, \text{rep}_\text{index}, \text{rep}_\text{B}) := \text{lift}'(\overline{\text{rep}_\text{A}}, \overline{\text{rep}_\text{B}})$
  - $\overline{\text{lift}}(\text{rep}_\text{A}, \text{rep}_\text{index}, \text{rep}_\text{B}) : \overline{\text{arg}_A}(\gamma, A_{ref}, B_{ref}) \rightarrow \text{Arg'}_\text{A}(\gamma, A, B)$
  - $\overline{\text{lift}}(\text{rep}_\text{A}, \text{rep}_\text{index}, \text{rep}_\text{B}) : \overline{\text{Arg'}_\text{A}}(\gamma, \overline{A_{ref}}, \overline{B_{ref}}) \rightarrow \text{Arg'}_\text{A}(\gamma, A, B)$

Representation for arguments

- $\mathsf{rep}_{\mathsf{A},1} := \overline{\mathsf{lift}}(\mathsf{rep}_{\mathsf{A}}, \mathsf{rep}_{\mathsf{index}}, \mathsf{rep}_{\mathsf{B}})$
- $\mathsf{rep}_{\mathsf{A},1} : \overline{arg_A}(\gamma, A_{ref}, B_{ref}) \to \mathsf{Arg'}_{\mathsf{A}}(\gamma, A, B)$
- We now have represenations for *arguments* to constructors

Example

Let's look at γ_{ext} again:

$$\text{extension} : ((p : \text{Platform}) \times \text{Building}(p)) \to \text{Platform}$$

$$\gamma_{ext} = \text{A-ind}(\mathbf{1}, \text{B-ind}(\mathbf{1}, \lambda * .\hat{p}, \text{nil}_{\text{A}}))$$

and

$$\text{Arg'}_{\text{A}}(\gamma_{ext}, \text{Platform}, \text{Building}) = (p : \mathbf{1} \to \text{Platform}) \times \mathbf{1} \to \text{Building}(p(*)) \times \mathbf{1}$$

$$\text{Arg'}_{\text{A}}(\gamma_{ext}, \text{Platform}, \text{Building}) = (p : \text{Platform}) \times \text{Building}(p) \times \mathbf{1}$$

Also assume we have $A_{ref} = B_{ref} = \mathbf{0} + \mathbf{1}$
Then $\overline{A_{ref}} = A_{ref} + B_{ref}$ has two elements: $\hat{p} = \text{inl}(\text{inr}(*))$ and $\widehat{pb} = \text{inr}(\text{inr}(*))$

Example

- $\overline{B_{ref}}(\hat{p}) = \mathbf{0}$
- $\overline{B_{ref}}(\widehat{pb}) = \mathbf{1}$
- $\widehat{\langle pb \rangle} = \langle \widehat{pb}, *, * \rangle$ is the only element in $\overline{\mathrm{arg}_A}(\gamma_{ext}, A_{ref}, B_{ref})$
- $\mathrm{rep}_{A,1}(\widehat{\langle pb \rangle}) = \langle \mathrm{rep}_{\mathrm{index}}(\widehat{pb}), \mathrm{rep}_{\mathrm{B}}(\widehat{pb}), * \rangle = \langle p, b, * \rangle$

Nested Constructors

Our arg fuction has given us the tools to go from a representation for A and B to represenations of arguments of constructors

Nested Constructors

Our arg fuction has given us the tools to go from a representation for A and B to represenations of arguments of constructors

Now, we want to be able to nest those constructors as well

Nested Constructors

Let's say we have a sequence $\vec{B}_{ref(n)} = B_{ref,0}, B_{ref,1}, ..., B_{ref,n-1}$. (Note that $\vec{B}_{ref(0)}$ is just an empty sequence.)
We now define:

$$\mathrm{arg}_A^0(\gamma, A_{ref}, \vec{B}_{ref(0)}) = A_{ref}$$

$$\mathrm{arg}^{n+1}0_A(\gamma, A_{ref}, \vec{B}_{ref(n+1)}) = \overline{\mathrm{arg}_A}(\gamma, \bigplus_{i=0}^{n} \mathrm{arg}_{\mathrm{A}}^i(\gamma, A_{ref}, \vec{B}_{ref(i)}), B_{ref,n})$$

$\mathrm{arg}_{\mathrm{A}}^k$ represents k nested constructors

Looking at $\mathrm{arg}_\mathrm{A}^1$

$$\begin{aligned}\mathrm{arg}_\mathrm{A}^1(\gamma, A_{ref}, \vec{B}_{ref(1)}) &= \overline{\mathrm{arg}_\mathrm{A}}(\gamma, \mathrm{arg}_\mathrm{A}^0(\gamma, A_{ref}, \vec{B}_{ref(0)}), B_{ref,0}) \\ &= \overline{\mathrm{arg}_\mathrm{A}^0}(\gamma, A_{ref}, B_{ref,0})\end{aligned}$$

In the "real" world

$$\mathrm{Arg}^0_\mathrm{A}(\gamma, A, \vec{B}_{(0)}) = A$$

$$\mathrm{Arg}^{n+1}_\mathrm{A}(\gamma, A_{ref}, \vec{B}_{n+1}) = \mathrm{Arg'}_\mathrm{A}(\gamma, \coprod_{i=0}^{n} \mathrm{Arg}^i_\mathrm{A}(\gamma, A, \vec{B}_{(i)}), \bigsqcup_{i=0}^{n} B_i)$$

Where $\vec{B}_{(n)} = B_0, B_1, ..., B_{n-1}$, with $B_i : \mathrm{Arg}^i_\mathrm{A}(\gamma_A, A, \vec{B}_{(i-1)}) \to \mathrm{Set}$

$\text{rep}_{\text{index, i}}$

If we now have the following:

- $\text{rep}_{\text{A}} : A_{ref} \to A$
- $\text{rep}_{\text{index, i}} : B_{ref,i} \to \text{Arg}^{i}_{\text{A}}(\gamma, A, \vec{B})$
- $\text{rep}_{\text{B,i}} : (x : B_{ref,i}) \to B_i(\text{rep}_{\text{index, i}}(x))$

Then we can construct:

- $\text{rep}_{\text{A, n}} : \text{arg}^{n}_{\text{A}}(\gamma, A_{ref}, \vec{B}_{ref}) \to \text{Arg}^{n}_{\text{A}}(\gamma, A, \vec{B})$
  - $\text{rep}_{\text{A, 0}} = \text{rep}_{\text{A}}$
  - $\text{rep}_{\text{A, n + 1}} = \overline{\text{lift}}(\|^{n}_{i=0}\text{rep}_{\text{A, i}}, \text{in}_n \circ \text{rep}_{\text{index, n}}, \text{rep}_{\text{B, n}})$

SP_B

- SP_B Codes for constructors
- Arg_B Maps codes on types
- $Index_B$ assigns elements $b : B(a)$ to their index a

Formation rule for SP_B

SP_B is like SP_A but two differences

- We can refer to constructors of A ($\gamma_A : SP'_A$ and $B_{\mathrm{ref}}, 0, \ldots B_{\mathrm{ref}}, i$)
- We need an index for codomain of constructor

Formation rule for SP_B

$$\frac{\gamma_A : \mathrm{SP}'_{\mathrm{A}} \quad A_{\mathrm{ref}} : \mathrm{Set} \quad B_{\mathrm{ref},\ 0}, B_{\mathrm{ref},\ 1}, \ldots, B_{\mathrm{ref},\ k} : \mathrm{Set}}{\mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\mathrm{ref}}, B_{\mathrm{ref},\ 0}, B_{\mathrm{ref},\ 1}, \ldots, B_{\mathrm{ref},\ k}) : \mathrm{Type}}$$

Formation rule for SP_B

$$\frac{A_{\text{ref}} : \text{Set} \qquad B_{\text{ref}} : \text{Set}}{\text{SP}_{\text{A}}(A_{\text{ref}}, B_{\text{ref}}) : \text{Type}}$$

$$\frac{\boxed{\gamma_A : \text{SP}'_{\text{A}}} \quad A_{\text{ref}} : \text{Set} \quad B_{\text{ref, 0}}, \boxed{B_{\text{ref, 1}}, \ldots, B_{\text{ref, }k}} : \text{Set}}{\text{SP}_{\text{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref, 0}}, B_{\text{ref, 1}}, \ldots, B_{\text{ref, }k}) : \text{Type}}$$

Formation rule for SP_B

$$\text{hangingUnder} : ((p : \text{Platform}) \times (b : \text{Building}(p))) \to \text{Building}(\underline{\text{extension}}(\langle p, b \rangle)).$$

$$\frac{\gamma_A : \text{SP}'_{\text{A}} \quad A_{\text{ref}} : \text{Set} \quad B_{\text{ref, }0}, \boxed{B_{\text{ref, }1}, \ldots, B_{\text{ref, }k}} : \text{Set}}{\text{SP}_{\text{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref, }0}, B_{\text{ref, }1}, \ldots, B_{\text{ref, }k}) : \text{Type}}$$

Introduction rules for SP_B

$\mathrm{nil}_{\mathrm{B}}(a_{\mathrm{index}})$

$\mathrm{nonind}(K, \gamma)$

$\text{A-ind}(K, \gamma) :$

$\mathrm{B}_{\ell}\text{-ind}(K, h_{\mathrm{index}}, \gamma)$

Introduction rules for SP_B

$$\frac{a_{\text{index}} : +_{i=0}^{k} \operatorname{arg}_{\mathrm{A}}^{i}(\gamma_A, A_{\text{ref}}, \vec{B}_{\text{ref}})}{\operatorname{nil}_{\mathrm{B}}(a_{\text{index}}) : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\ 0}, \ldots, B_{\text{ref},\ k})}$$

$$\frac{K : \mathrm{Set} \quad \gamma : K \to \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\ 0}, \ldots, B_{\text{ref},\ k})}{\operatorname{nonind}(K, \gamma) : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\ 0}, \ldots, B_{\text{ref},\ k})}$$

$$\frac{K : \mathrm{Set} \quad \gamma : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}} + K, B_{\text{ref},\ 0}, \ldots, B_{\text{ref},\ k})}{\text{A-ind}(K, \gamma) : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\ 0}, \ldots, B_{\text{ref},\ k})}$$

$$\frac{\begin{array}{c} h_{\text{index}} : K \to \operatorname{arg}_{\mathrm{A}}^{\ell}(\gamma_A, A_{\text{ref}}, \vec{B}_{\text{ref}}) \\ K : \mathrm{Set} \quad \gamma : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\ 0}, \ldots, B_{\text{ref},\ \ell} + K, \ldots, B_{\text{ref},\ k}) \end{array}}{\mathrm{B}_{\ell}\text{-ind}(K, h_{\text{index}}, \gamma) : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\ 0}, \ldots, B_{\text{ref},\ k})}$$

Introduction rules for SP_B

$$\frac{a_{\text{index}} : +_{i=0}^{k} \operatorname{arg}_{\mathrm{A}}^{i}(\gamma_A, A_{\text{ref}}, \vec{B}_{\text{ref}})}{\mathrm{nil}_{\mathrm{B}}(a_{\text{index}}) : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\,0}, \ldots, B_{\text{ref},\,k})}$$

$$\frac{}{\mathrm{nil}_{\mathrm{A}} : \mathrm{SP}_{\mathrm{A}}(A_{\text{ref}}, B_{\text{ref}})}$$

$$\frac{K : \mathrm{Set} \quad \gamma : K \to \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\,0}, \ldots, B_{\text{ref},\,k})}{\mathrm{nonind}(K, \gamma) : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\,0}, \ldots, B_{\text{ref},\,k})}$$

$$\frac{K : \mathrm{Set} \quad \gamma : K \to \mathrm{SP}_{\mathrm{A}}(A_{\text{ref}}, B_{\text{ref}})}{\mathrm{nonind}(K, \gamma) : \mathrm{SP}_{\mathrm{A}}(A_{\text{ref}}, B_{\text{ref}})}$$

$$\frac{K : \mathrm{Set} \quad \gamma : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}} + K, B_{\text{ref},\,0}, \ldots, B_{\text{ref},\,k})}{\text{A-ind}(K, \gamma) : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\,0}, \ldots, B_{\text{ref},\,k})}$$

$$\frac{K : \mathrm{Set} \quad \gamma : \mathrm{SP}_{\mathrm{A}}(A_{\text{ref}} + K, B_{\text{ref}})}{\text{A-ind}(K, \gamma) : \mathrm{SP}_{\mathrm{A}}(A_{\text{ref}}, B_{\text{ref}})}$$

$$\frac{\begin{array}{c} h_{\text{index}} : K \to \operatorname{arg}_{\mathrm{A}}^{\ell}(\gamma_A, A_{\text{ref}}, \vec{B}_{\text{ref}}) \\ K : \mathrm{Set} \quad \gamma : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\,0}, \ldots, B_{\text{ref},\,\ell} + K, \ldots, B_{\text{ref},\,k}) \end{array}}{\mathrm{B}_{\ell}\text{-ind}(K, h_{\text{index}}, \gamma) : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\,0}, \ldots, B_{\text{ref},\,k})}$$

$$\frac{K : \mathrm{Set} \quad h_{\text{index}} : K \to A_{\text{ref}} \quad \gamma : \mathrm{SP}_{\mathrm{A}}(A_{\text{ref}}, B_{\text{ref}} + K)}{\text{B-ind}(K, h_{\text{index}}, \gamma) : \mathrm{SP}_{\mathrm{A}}(A_{\text{ref}}, B_{\text{ref}})}$$

Introduction rules for SP_B

$$\frac{a_{\text{index}} : +_{i=0}^{k} \operatorname{arg}_{\mathrm{A}}^{i}(\gamma_A, A_{\text{ref}}, \vec{B}_{\text{ref}})}{\operatorname{nil}_{\mathrm{B}}(a_{\text{index}}) : \mathrm{SP}_{\mathrm{B}}(\gamma_A, A_{\text{ref}}, B_{\text{ref. }0}, \ldots, B_{\text{ref. }k})}$$

$$\frac{}{\operatorname{nil}_{\mathrm{A}} : \mathrm{SP}_{\mathrm{A}}(A_{\text{ref}}, B_{\text{ref}})}$$

Introduction rules for SP_B

$$\frac{K : \mathrm{Set} \quad \gamma : \mathrm{SP_B}(\gamma_A, A_{\mathrm{ref}}, B_{\mathrm{ref},\ 0}, \ldots, B_{\mathrm{ref},\ \ell} + K, \ldots, B_{\mathrm{ref},\ k}) \quad h_{\mathrm{index}} : K \to \mathrm{arg}^{\ell}_{\mathrm{A}}(\gamma_A, A_{\mathrm{ref}}, \vec{B}_{\mathrm{ref}})}{\mathrm{B}_{\ell}\text{-ind}(K, h_{\mathrm{index}}, \gamma) : \mathrm{SP_B}(\gamma_A, A_{\mathrm{ref}}, B_{\mathrm{ref},\ 0}, \ldots, B_{\mathrm{ref},\ k})}$$

$$\frac{K : \mathrm{Set} \quad h_{\mathrm{index}} : K \to A_{\mathrm{ref}} \quad \gamma : \mathrm{SP_A}(A_{\mathrm{ref}}, B_{\mathrm{ref}} + K)}{\mathrm{B}\text{-ind}(K, h_{\mathrm{index}}, \gamma) : \mathrm{SP_A}(A_{\mathrm{ref}}, B_{\mathrm{ref}})}$$

Introduction rules for SP_B

$$\frac{K : \mathrm{Set} \quad \dfrac{h_{\text{index}} : K \to \boxed{\mathrm{arg}^{\ell}_{\mathrm{A}}(\gamma_A, A_{\text{ref}}, \vec{B}_{\text{ref}})}}{\gamma : \mathrm{SP_B}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\,0}, \ldots, B_{\text{ref},\,\ell} + K, \ldots, B_{\text{ref},\,k})}}{\mathrm{B}_\ell\text{-ind}(K, h_{\text{index}}, \gamma) : \mathrm{SP_B}(\gamma_A, A_{\text{ref}}, B_{\text{ref},\,0}, \ldots, B_{\text{ref},\,k})}$$

$$\frac{K : \mathrm{Set} \quad \dfrac{h_{\text{index}} : K \to \boxed{A_{\text{ref}}}}{\gamma : \mathrm{SP_A}(A_{\text{ref}}, B_{\text{ref}} + K)}}{\mathrm{B}\text{-ind}(K, h_{\text{index}}, \gamma) : \mathrm{SP_A}(A_{\text{ref}}, B_{\text{ref}})}$$

Arg_B

nil_b, nonind, A-ind are analogous to Arg_A

$$\begin{aligned} \text{nil}_B(a_\text{index}) &\rightarrow \mathbf{1} \\ \text{nonind}(K, \gamma) &\rightarrow (k : K) \times \text{recursive call} \\ \text{A-ind}(K, \gamma) &\rightarrow (j : K \rightarrow A) \times \text{recursive call} \end{aligned}$$

Arg_B

$$\mathrm{B}_I\text{-ind}(K, h_{index}, \gamma) \rightarrow$$
$$(j : (k : K) \rightarrow B_I((Rep_{A,I} \circ h_{index}(k))) \times \text{ recursive call}$$

The last missing piece is now Index_B

Again we do case distinction on the codes

$$\mathrm{Index_B}(\gamma_A, A_{\mathrm{ref}}, \vec{B}_{\mathrm{ref}}, \underline{\mathrm{nil_B}(a_{\mathrm{index}})}, A, \vec{B}, \mathrm{rep_A}, \vec{\mathrm{rep}}_{\mathrm{index}}, \vec{\mathrm{rep}}_{\mathrm{B}}, \star) = (\coprod_{i=0}^{k} \mathrm{rep}_{\mathrm{A},i})(a_{\mathrm{index}})$$

$$\mathrm{Index_B}(\gamma_A, A_{\mathrm{ref}}, \vec{B}_{\mathrm{ref}}, \underline{\mathrm{nonind}(K,\gamma)}, A, \vec{B}, \mathrm{rep_A}, \vec{\mathrm{rep}}_{\mathrm{index}}, \vec{\mathrm{rep}}_{\mathrm{B}}, \langle \underline{k,} y\rangle) = \mathrm{Index_B}(\_, \_, \_\_, \gamma(k), \_, \_\_, \_, \_\_, \_\_, y)$$

$$\mathrm{Index}_\mathrm{B}(\gamma_A, A_\mathrm{ref}, \vec{B}_\mathrm{ref}, \underline{\text{A-ind}(K,\gamma)}, A, \vec{B}, \mathrm{rep}_\mathrm{A}, \vec{\mathrm{rep}}_\mathrm{index}, \vec{\mathrm{rep}}_\mathrm{B}, \langle j, y\rangle) = \mathrm{Index}_\mathrm{B}(\_, A_\mathrm{ref} + K, \_\_, \gamma, \_, \_\_, \mathrm{rep}_\mathrm{A} \sqcup j, \_\_, \_\_, y)$$

$$\mathrm{Index_B}(\gamma_A, A_{\mathrm{ref}}, \vec{B}_{\mathrm{ref}}, \underline{\mathrm{B}_n\text{-ind}(K, h, \gamma)}, A, \vec{B}, \mathrm{rep_A}, \vec{\mathrm{rep}}_{\mathrm{index}}, \vec{\mathrm{rep}}_{\mathrm{B}}, \langle j, y\rangle) =$$

$$\mathrm{Index_B}(\_, \_, \_\_, B_{\mathrm{ref},\ n}{+}K, \_\_, \gamma, \_, \_\_, \_, \_\_, \mathrm{rep}_{\mathrm{index},n} \sqcup (\mathrm{rep}_{\mathrm{A},n} \circ h), \_\_, \_\_, \mathrm{rep}_{\mathrm{B},n} \sqcup j, \_\_, y).$$

Formation rules

$$\frac{\gamma_A : \mathrm{SP'}_{\mathrm{A}} \qquad \gamma_B : \mathrm{SP'}_{\mathrm{B}}(\gamma_A)}{A_{\gamma_A,\gamma_B} : \mathrm{Set}}$$

$$\frac{\gamma_A : \mathrm{SP'}_{\mathrm{A}} \qquad \gamma_B : \mathrm{SP'}_{\mathrm{B}}(\gamma_A)}{B_{\gamma_A,\gamma_B} : A_{\gamma_A,\gamma_B} \rightarrow \mathrm{Set}}$$

All rules will have the premises $\gamma_A : \mathrm{SP'}_{\mathrm{A}}$ and $\gamma_B : \mathrm{SP'}_{\mathrm{B}}(\gamma_A)$, so from now on we'll leave them out

Introduction rule for A

$$\frac{a : \mathrm{Arg'}_\mathrm{A}(\gamma_A, A_{\gamma_A,\gamma_B}, B_{\gamma_A,\gamma_B})}{\mathrm{intro}_A(a) : A_{\gamma_A,\gamma_B}}$$

Introduction Rule for B

$$\frac{b : \text{Arg'}_{\text{B}}(\gamma_A, A_{\gamma_A,\gamma_B}, B_{\gamma_A,\gamma_B}, B_1, \ldots, B_k)}{\text{intro}_B(b) : B_{\gamma_A,\gamma_B}(\overline{\text{index}}(b))}$$

Introduction Rule for B

$$\frac{b : \text{Arg'}_{\text{B}}(\gamma_A, A_{\gamma_A,\gamma_B}, B_{\gamma_A,\gamma_B}, \color{red}{B_1, \ldots, B_k})}{\text{intro}_B(b) : B_{\gamma_A,\gamma_B}(\overline{\text{index}}(b))}$$

We don't have these yet!

B_i's

We still need the various functions $B_i : \mathrm{Arg}^i_\mathrm{B}(\gamma_A, A_{\gamma_A,\gamma_B}, B_{\gamma_A,\gamma_B}) \to \mathrm{Set}$
We will need to define:

$$\mathrm{intro}_n : \mathrm{Arg}^n_\mathrm{A}(\gamma_A, A_{\gamma_A,\gamma_B}, B_0, ..., B_{n-1}) \to A_{\gamma_A,\gamma_B}$$
$$B_n : \mathrm{Arg}^n_\mathrm{A}(\gamma_A, A_{\gamma_A,\gamma_B}, B_0, ..., B_{n-1}) \to \mathrm{Set}$$

B_i's

$$\text{intro}_0 = \text{id}$$

$$\text{intro}_{n+1} = \text{intro}_A \circ \text{lift}'\left(\bigsqcup_{i=0}^{n} \text{intro}_i, \bigsqcup_{i=0}^{n} (\lambda a.id)\right)$$

$$B_i(x) = B_{\gamma_A,\gamma_B}(\text{intro}_i(x))$$

One more definition

$$\overline{index} = \left(\bigsqcup_{i=0}^{k} \text{intro}_i \right) \circ \text{Index}'_B(\gamma_A, \gamma_B, A_{\gamma_A,\gamma_B}, B_0, \ldots, B_k)$$

Introduction Rule for B

$$\frac{b : \text{Arg'}_{\text{B}}(\gamma_A, A_{\gamma_A,\gamma_B}, B_{\gamma_A,\gamma_B}, B_1, ..., B_k)}{\text{intro}_B(b) : B_{\gamma_A,\gamma_B}(\overline{\text{index}}(b))}$$

Questions?